

# Conductance scaling of junctions of Luttinger-liquid wires out of equilibrium

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We develop the renormalization group theory of the conductances of N-lead junctions of spinless Luttinger-liquid wires as functions of bias voltages applied to N independent Fermi-liquid reservoirs. Based on the perturbative results up to second order in the interaction we demonstrate that the conductances obey scaling. The corresponding renormalization group  $\beta$ -functions are derived up to second order.

## I. INTRODUCTION

The charge transport through junctions connecting quantum wires modeled by the Tomonaga-Luttinger liquid model (TLL) has been studied intensely over the past several decades. In the linear response regime it has been shown since the first studies of a two-lead junction<sup>1-4</sup> that the conductance obeys scaling as a function of temperature, at least in the vicinity of certain special values of the conductance. This behavior is captured in the framework of a renormalization group (RG) formulation, where the special values are identified as fixed points of the RG flow. Initially the flow equations were derived within the bosonization approach. The latter has the difficulty that the conductance of a wire of finite length depends on the contact resistances at the links to the external charge reservoirs (accounting for the transition of bosonic excitations into fermionic quasiparticles), which so far have not been determined. Alternatively, a purely fermionic formulation may be used, which avoids the problem of contact resistance.

The latter approach has been pioneered by<sup>5</sup> in the limit of weak interaction and was later extended to arbitrary coupling strength by<sup>6</sup>. In simple words, the standard procedure for how to derive the RG flow equations, e.g. for the two-lead junction, is to first calculate the conductance  $G_0(\theta)$  in the absence of interaction, as a function of the parameter  $\theta$  (or of several parameters in the case of multi-lead junctions) determining the scattering strength of the junction. Then the conductance is calculated in perturbation theory in the interaction, allowing to identify the linear logarithmic corrections (for example  $\propto \ln(\omega_0/T)$ , if the temperature  $T$  is cutting off the infrared singularities of the theory and  $\omega_0$  is an ultraviolet cutoff), and extract the RG  $\beta$ -function as  $\beta = -dG/d\ln T$  at  $\ln(\omega_0/T) = 0$ . The resulting function  $\beta$  depends on the interaction strength  $\alpha$  and on the parameter  $\theta$  characterizing the junction. Inverting the functional dependence  $G_0(\theta)$ , the parameter  $\theta$  may be expressed by  $G_0$ , which in the sense of the RG structure may be replaced by the renormalized  $G$  at scale  $T$ . This procedure may be justified within a more rigorous scheme using ideas first formulated by<sup>7,8</sup>, see<sup>6</sup>. In order to explicitly demonstrate the scaling property, it should

be shown that all terms of powers higher than linear in  $\ln(\omega_0/T)$ , appearing in perturbation theory, are generated by iteration of the RG equations. For the case of the conductance of a two-lead junction this has been verified up to order  $(\ln(\omega_0/T))^{36}$ . The approach sketched above has also been applied to multilead junctions, such as the Y-junctions in the weak coupling limit<sup>9-11</sup> and at strong coupling<sup>12</sup>, and chiral Y-junctions<sup>13-15</sup>, as well as X-junctions<sup>16</sup>.

The results on transport through Y-junctions obtained in our previous work are generally in good agreement with results obtained by the bosonization method (BM) in the linear response regime<sup>17-20</sup>, keeping in mind that the overall magnitude of the current can not be determined accurately by BM, as mentioned above. In the few cases where discrepancies have arisen, such as in the limit of very strong attractive interaction,<sup>12</sup> the BM calculation employed additional assumptions which we believe to be incorrect. There are only few works on transport through Y-junctions out of equilibrium, in which scaling has been assumed to exist without proof<sup>21,22</sup>.

The fermionic transport formulation is general and physically appealing. It may be extended to systems out of equilibrium in a natural way. Transport through a two-lead junction at finite voltage bias  $V$  and low temperatures has been considered in<sup>23</sup>, with results in agreement with those of other methods, where applicable<sup>24-27</sup>. Not too surprisingly, one finds the conductance following a power law in  $V$ , with exponent identical to the one governing the temperature power law of the linear response conductance. Recently, transport through a Y-junction out of equilibrium has been studied by assuming the scaling property to hold in this case as well<sup>22</sup>. The latter assumption is not trivial, if only for the following reason: the Y-junction is connected to three charge reservoirs held at three different chemical potentials, in general. Consequently, the two independent conductances depend on two independent bias voltages  $V_a, V_b$ . There appears to be scaling of the conductances in both variables,  $V_a$  and  $V_b$ . The question is then how the scaling in  $T$  is expressed in terms of voltage, i.e. which of the two voltages or both enter the corresponding formulas. Indeed it has been found in<sup>22</sup> that the scaling as a function of  $V_a, V_b$  may not be obtained from the scaling of the

linear response conductance with  $T$  by any simple recipe.

In the present paper we demonstrate that scaling in the case of multi-lead junctions out of equilibrium is valid, by explicitly calculating the terms of second power in the scaling variable  $\Lambda = \ln(\omega_0/\epsilon)$  in the conductances of a symmetric Y-junction in second order in the interaction. Here  $\omega_0$  and  $\epsilon$  are ultraviolet and infrared cutoffs in energy. We then show that all of these terms are generated by the RG equations, proving the validity of scaling, at least up to this order.

## II. THE MODEL

We consider a system of spinless fermions in one dimension, interacting in each of  $N$  quantum wires in the region  $a < |x_j| < L$ ,  $j = 1, \dots, N$ , adiabatically connected to  $N$  charge reservoirs at  $|x_j| > L$ . The  $N$  wires are connected by a junction located in the narrow regime  $|x_j| < a$ , which scatters the fermions as described by an  $S$  matrix with elements  $S_{jk}$ , where  $j, k = 1, \dots, N$ .

We assume the interaction to be described by a TLL model in the form

$$H = \int_0^\infty dx \sum_{j=1}^N [H_j^0(x) + H_j^{int}(x)\Theta(x; a, L)] \quad (1)$$

where

$$H_j^0(x) = v_j [\psi_{j,in}^\dagger(x) i\nabla \psi_{j,in}(x) - \psi_{j,out}^\dagger(x) i\nabla \psi_{j,out}(x)]$$

and

$$H_j^{int}(x) = 2\pi v_j \alpha_j [\psi_{j,in}^\dagger(x) \psi_{j,in}(x) \psi_{j,out}^\dagger(x) \psi_{j,out}(x)]$$

Here  $v_j$  is the Fermi velocity,  $\alpha_j$  is the interaction constant in lead  $j$  and  $\Theta(x; a, L) = 1$  in the interval  $a < |x| < L$  and zero elsewhere. The fermionic field operators  $\psi_{j,\eta_j}^\dagger(x)$  create particles at position  $x$  in scattering states  $|j, \eta_j; \omega\rangle$  of energy  $\omega$ , in wire  $j$  and with chirality  $\eta_j = \pm 1$ , labeling incoming ( $\eta_j = -1$ ) and outgoing ( $\eta_j = +1$ ) states. In the following we will put  $v_j = 1$  for simplicity. The outgoing fermion operators are connected with the incoming ones by the  $S$  matrix,  $\psi_{j,out}(0) = S_{jk} \psi_{k,in}(0)$ .

The perturbation theory will be formulated in the language of Keldysh matrix single particle Green's functions

$$\underline{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \quad (2)$$

in the non-interacting limit. The Green's functions depend on energy  $\omega$ , position  $x$ , wire index  $j$  and chirality  $\eta_j$ ,  $\underline{G} = \underline{G}_\omega(l_{\eta_l}, y; j_{\eta_j}, x)$ .

The Green's functions for each pair of chiralities  $\eta_l, \eta_j$  are given by

$$\begin{aligned} \underline{G}_\omega(l_+, y; j_+, x) &= -ie^{i\omega(y-x)} \begin{bmatrix} \theta(y-x)\delta_{lj}, & S_{lm}S_{jm}^*h_m \\ 0, & -\theta(x-y)\delta_{lj} \end{bmatrix}, \\ \underline{G}_\omega(l_+, y; j_-, x) &= -ie^{i\omega(y+x)} \begin{bmatrix} S_{lj}, & S_{lj}h_j \\ 0, & 0 \end{bmatrix}, \\ \underline{G}_\omega(l_-, y; j_+, x) &= -ie^{-i\omega(y+x)} \begin{bmatrix} 0, & S_{jl}^*h_l \\ 0, & -S_{jl}^* \end{bmatrix}, \\ \underline{G}_\omega(l_-, y; j_-, x) &= -ie^{i\omega(x-y)}\delta_{lj} \begin{bmatrix} \theta(x-y), & h_l \\ 0, & -\theta(y-x) \end{bmatrix}, \end{aligned} \quad (3)$$

where  $h_l(\omega) = \tanh[(\omega - \mu_l)/2T]$  is the Keldysh function, with  $\mu_l$  the chemical potential in wire  $l$ ; summation over  $m$  is implied in the first line of (3). The functions  $h_l(\omega)$  carry the information on the out of equilibrium conditions.

## III. DERIVATION OF RENORMALIZATION GROUP EQUATIONS FOR THE CONDUCTANCES

We consider the conductances  $G_j$ ,  $j = 1, \dots, N_G$  of a  $N$ -lead junction out of equilibrium (here  $N_G < N^2$  is the number of independent conductances). In order to show that the conductances obey scaling, we follow the reasoning first developed by Callan and Symanczik Ref. 7,8 and apply it to the problem at hand. Assume that we know the perturbation series of the conductances  $G_j$ . The conductances depend on the scattering properties of the junction, expressed by the S-matrix elements  $S_{lj}$  (which may be expressed in terms of the conductances  $G_j^0$  of the system in the absence of interactions), and on the interaction (coupling constant  $\alpha$ ). The possible existence of scaling is signaled by the appearance of powers of the scaling variable  $\Lambda = \ln(\omega_0/\epsilon)$ . The perturbation series of  $G_j$  in terms of the interaction has the general form

$$G_j = G_j^0 + \alpha A_j(\{G_i^0\}) + \alpha^2 B_j(\{G_i^0\}) + O(\alpha^3) \quad (4)$$

where  $A_j, B_j$  are polynomials of first and second order in the scaling variable  $\Lambda$ , respectively, polynomials of the bare conductances  $G_j^0$ , functions of the  $N_V$  independent bias voltages  $V_j$ , and of additional coupling constants  $\alpha_j$  in the form of ratios  $\alpha_j/\alpha$ . Now we invert the series to express the conductances  $G_j^0$  in the non-interacting limit in terms of the full conductances  $G_j$

$$G_j^0 = G_j + \alpha \bar{A}_j(\{G_i\}) + \alpha^2 \bar{B}_j(\{G_i\}) + O(\alpha^3) \quad (5)$$

where  $\bar{A}_j, \bar{B}_j$  are again polynomials of first and second order in  $\Lambda$ . By substituting  $G_j^0$  as given in Eq. (5) into Eq. (4) we find the following relations

$$\begin{aligned} \bar{A}_j(\{G_i\}) &= -A_j(\{G_i\}) \\ \bar{B}_j(\{G_i\}) &= -B_j(\{G_i\}) - \sum_l \frac{\partial A_j(\{G_i\})}{\partial G_l} \bar{A}_l(\{G_i\}) \end{aligned} \quad (6)$$

We now use that  $G_j^0(\{G_i\})$  must be independent of  $\Lambda$

$$\frac{dG_j^0(\{G_i\})}{d\Lambda} = 0 = \frac{\partial G_j^0(\{G_i\})}{\partial \Lambda} + \sum_l \frac{\partial G_j^0(\{G_i\})}{\partial G_l} \frac{\partial G_l}{\partial \Lambda} \quad (7)$$

to find the renormalization group equation

$$\frac{\partial G_j}{\partial \Lambda} = - \sum_l \left( \frac{\partial G_j^0(\{G_i\})}{\partial G_l} \right)^{-1} \frac{\partial G_l^0(\{G_i\})}{\partial \Lambda} \quad (8)$$

The inversion of the matrix  $D_{jl} = \partial G_j^0(\{G_i\})/\partial G_l$  is obtained from Eq. (5) as

$$\left( \frac{\partial G_j^0(\{G_i\})}{\partial G_l} \right)^{-1} = \delta_{jl} + \alpha \frac{\partial A_j(\{G_i\})}{\partial G_l} \quad (9)$$

Substituting  $\partial G_l^0(\{G_i\})/\partial \Lambda$  we finally get

$$\begin{aligned} \frac{\partial G_j}{\partial \Lambda} &= \alpha \frac{\partial A_j(\{G_i\})}{\partial \Lambda} \\ &+ \alpha^2 \left( \frac{\partial B_j(\{G_i\})}{\partial \Lambda} - \sum_l \frac{\partial^2 A_j(\{G_i\})}{\partial \Lambda \partial G_l} A_l(\{G_i\}) \right) \end{aligned} \quad (10)$$

In order for scaling to hold, the r.h.s. of Eq. (10) should not depend on  $\Lambda$ . More specifically, it should be  $\mathcal{O}(\Lambda^0)$ , possibly containing Heaviside step functions (see below), indicating a stop of the RG flow. Verifying this property amounts to proving the validity of scaling up to the order considered.

#### IV. SYMMETRIC Y-JUNCTION OUT OF EQUILIBRIUM

We now apply the above general derivation of RG equations for the conductances to a concrete example. We consider charge transport through a Y-junction with a symmetric main wire (labelled 1, 2) contacted by a tip wire at the center (3) (see Fig.1). The three half-wires of length  $L$  are adiabatically connected with reservoirs kept at chemical potentials  $\mu_j$ ,  $j = 1, 2, 3$ . We assume that there is no interaction within the junction of radius  $a$ . The scattering states of each wire are labeled by wire index  $j$ , chirality  $\eta_j = +, -$  (outgoing or ingoing), energy  $\omega$ , and position  $x > 0$  in the interval  $[a, L]$ . The junction is symmetric in the sense that the interaction constants  $\alpha_1$  and  $\alpha_2$  in arms 1 and 2, respectively, are equal to each other,  $\alpha_1 = \alpha_2 \equiv \alpha$ . The third arm of the junction is a tunneling-tip wire, with interaction constant  $\alpha_3$ , which we will assume to vanish in the following. We define currents  $J_j$  flowing from the reservoirs toward the junction.

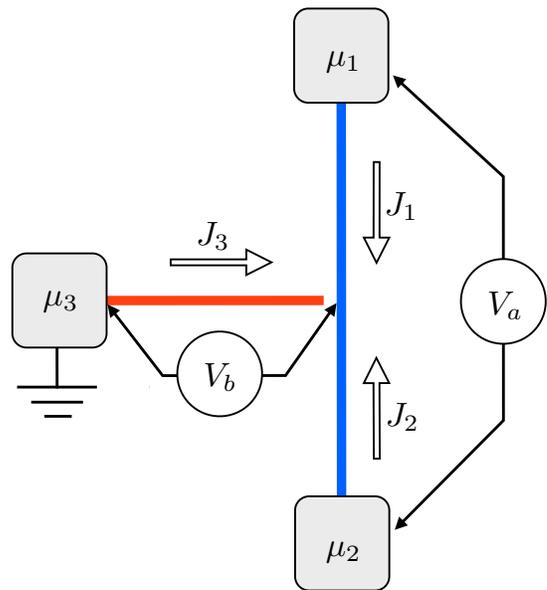


FIG. 1: Setup of the Y-junction out of equilibrium. The main wire is shown as a blue vertical line, the tunneling tip as a red horizontal line. The reservoirs at the chemical potentials  $\mu_{a,b,c}$  are depicted as gray blocks, with the currents  $J_{a,b,c}$  flowing out from them in the presence of the bias voltages  $V_a$  and  $V_b$ .

The  $S$  matrix of the symmetric Y-junction may be parametrized as follows

$$S = \begin{pmatrix} r_1 & t_1 & t_2 \\ t_1 & r_1 & t_2 \\ t_2 & t_2 & r_2 \end{pmatrix}. \quad (11)$$

The symmetric form of interaction,  $\alpha_1 = \alpha_2$ , keeps the renormalized  $S$  matrix in symmetric form (11). We use the parametrization

$$\begin{aligned} r_1 &= \frac{1}{2}(\cos \vartheta + e^{-i\psi}), & t_1 &= \frac{1}{2}(\cos \vartheta - e^{-i\psi}), \\ t_2 &= \frac{i}{\sqrt{2}} \sin \vartheta, & r_2 &= \cos \vartheta. \end{aligned} \quad (12)$$

It is convenient to introduce two independent currents  $J_{a,b}$  and two independent bias voltages  $V_{a,b}$  as follows:

$$J_a = \frac{1}{2}(J_1 - J_2), \quad V_a = \mu_1 - \mu_2 \quad (13)$$

for the main wire and

$$J_b = \frac{1}{3}(J_1 + J_2 - 2J_3) = -J_3, \quad V_b = \frac{1}{2}(\mu_1 + \mu_2) \quad (14)$$

for the tunneling tip. The conductances  $G$  are then defined as

$$J_a = G_a V_a + G_{ab} V_b, \quad J_b = G_{ba} V_a + G_b V_b \quad (15)$$

It is found that in the symmetric setup  $G_{ab}$  and  $G_{ba}$  appear due to asymmetry produced by the voltages, they

may be expressed in terms of the diagonal conductances  $G_a, G_b$  and therefore do not flow independently. We therefore do not consider the off-diagonal conductances in the following. In terms of parametrization (12) the conductances are given by  $G_a = (1 - \cos \vartheta \cos \psi)/2$ ,  $G_b = \sin^2 \vartheta$ .

### A. Perturbation theory results and RG-equation in first order

As shown in<sup>22</sup> the diagonal conductances in first order are given by

$$\begin{aligned} G_a(\epsilon) &= G_a^0 + \alpha (a_1^0 \Lambda_a + a_2^0 (\Lambda_{b+} + \Lambda_{b-})), \\ G_b(\epsilon) &= G_b^0 + \alpha a_3^0 (\Lambda_{b+} + \Lambda_{b-}) \end{aligned} \quad (16)$$

where  $G_{a,b}^0 = G_{a,b}(\omega_0)$  and we use a shorthand notation  $\Lambda_a = \ln(\omega_0/\max[\epsilon, V_a])$ ,  $\Lambda_{b\pm} = \ln(\omega_0/\max[\epsilon, |V_{b\pm}|])$ , with  $V_{b\pm} = V_b \pm V_a/2$ . Here we defined  $a_j^0 = a_j(G_a^0, G_b^0)$  with

$$\begin{aligned} a_1(G_a, G_b) &= -2G_a(1 - G_a) + \frac{1}{2}G_b, \\ a_2(G_a, G_b) &= -\frac{1}{8}[1 - G_a + g_3(1 - 2G_a)]G_b, \\ a_3(G_a, G_b) &= -\frac{1}{2}[1 - G_a - \frac{1}{2}G_b + g_3(1 - G_b)]G_b. \end{aligned} \quad (17)$$

and  $g_3 = \alpha_3/\alpha$ .

Differentiating (16) with respect to  $\Lambda = \ln(\omega_0/\epsilon)$  and replacing  $a_j^0 \rightarrow a_j$ , we obtain the RG  $\beta$ -functions as

$$\begin{aligned} \frac{\partial G_a}{\partial \Lambda} &= \beta_a(G_a, G_b) \\ &= \alpha [a_1(G_a, G_b) \theta_a(\epsilon) + a_2(G_a, G_b) \theta_+(\epsilon)], \\ \frac{\partial G_b}{\partial \Lambda} &= \beta_b(G_a, G_b) = \alpha a_3(G_a, G_b) \theta_+(\epsilon). \end{aligned} \quad (18)$$

with  $\theta_a(\epsilon) = \theta(\epsilon - V_a)$  and

$$\theta_+(\epsilon) = \theta(\epsilon - |V_{b-}|) + \theta(\epsilon - V_{b+}),$$

The effect of the  $\theta$ -functions is to define different forms of the functions  $A_{a,b}$  in different intervals of  $\epsilon$ , and hence  $\Lambda$ . For example, for given  $V_a > 2V_b > 0$  we have  $\theta_+(\epsilon) = 2$  for  $\epsilon > V_b + V_a/2$  and  $\theta_a(\epsilon) = 1$  for  $\epsilon > V_a$  (interval *I*),  $\epsilon > V_b + V_a/2$  and  $\theta_a(\epsilon) = 0$  for  $\epsilon < V_a$  (interval *II*),  $\theta_+(\epsilon) = 1$  for  $|V_b - V_a/2| < \epsilon < V_b + V_a/2$  (interval *III*) and  $\theta_+(\epsilon) = 0$  for  $|V_b - V_a/2| > \epsilon$  (interval *IV*).

If the differential equations (18) are valid, then the calculated second order corrections,  $\sim \alpha^2 \Lambda^2$ , are cancelled in the above procedure, leading to Eq. (10). Alternatively, we may compare these corrections with the predicted form stemming from equation (18).

The conductances  $G_{a,b}$  up to second order in  $\Lambda$  are determined by solving the RG-equations iteratively. We substitute the first order results, Eq. (16), into the  $\beta$ -functions and integrate. We get

$$G_{a,b} = G_{a,b}^0 + \alpha G'_{a,b} + \frac{1}{2} \alpha^2 G''_{a,b},$$

where for  $n = a, b$

$$\begin{aligned} G'_n &= \int_{\omega_0}^{\epsilon} \frac{d\epsilon'}{\epsilon'} \beta_n(G_a^0, G_b^0)_{\epsilon'} \\ G''_n &= 2 \int_{\omega_0}^{\epsilon} \frac{d\epsilon'}{\epsilon'} \int_{\omega_0}^{\epsilon'} \frac{d\epsilon''}{\epsilon''} \left[ \frac{\partial \beta_n(G_a^0, G_b^0)_{\epsilon'}}{\partial G_a^0} \beta_a(G_a^0, G_b^0)_{\epsilon''} \right. \\ &\quad \left. + \frac{\partial \beta_n(G_a^0, G_b^0)_{\epsilon'}}{\partial G_b^0} \beta_b(G_a^0, G_b^0)_{\epsilon''} \right] \end{aligned} \quad (19)$$

The expression in square brackets here reads

$$\begin{aligned} G''_a &= \frac{\partial a_1^0}{\partial G_a^0} a_1^0 \theta_a(\epsilon'') \theta_a(\epsilon') + \frac{\partial a_2^0}{\partial G_a^0} a_1^0 \theta_+(\epsilon') \theta_a(\epsilon'') \\ &\quad + \left( \frac{\partial a_1^0}{\partial G_a^0} a_2^0 + \frac{\partial a_1^0}{\partial G_b^0} a_3^0 \right) \theta_a(\epsilon') \theta_+(\epsilon'') \\ &\quad + \left( \frac{\partial a_2^0}{\partial G_a^0} a_2^0 + \frac{\partial a_2^0}{\partial G_b^0} a_3^0 \right) \theta_+(\epsilon') \theta_+(\epsilon'') \end{aligned} \quad (20)$$

and

$$\begin{aligned} G''_b &= \frac{\partial a_3^0}{\partial G_a^0} a_1^0 \theta_+(\epsilon') \theta_a(\epsilon'') \\ &\quad + \left( \frac{\partial a_3^0}{\partial G_a^0} a_2^0 + \frac{\partial a_3^0}{\partial G_b^0} a_3^0 \right) \theta_+(\epsilon') \theta_+(\epsilon'') \end{aligned} \quad (21)$$

Eqs. (20), (21) contain  $\theta$  functions in four different combinations. One of them,  $\theta_a(\epsilon') \theta_a(\epsilon'')$  results simply in  $\Lambda_a^2/2$ . Others, e.g.  $\theta_+(\epsilon') \theta_a(\epsilon'')$  lead to more complicated expressions and depend on the relation between  $V_a, V_b$ . As was shown in<sup>22</sup>, there are two most interesting cases. In one of them, with  $V_a \gtrsim V_b$ , we can let  $\theta_+(\epsilon) \simeq 2\theta(\epsilon - V_a)$ , which results in unique value of logarithm,  $\Lambda_a$ . In another regime,  $V_a \ll V_b$ , we let  $\theta_+(\epsilon) \simeq 2\theta(\epsilon - V_b)$  and obtain two values,  $V_a$  and  $V_b$ .

In this second regime we can express the integrals appearing in (19) as

$$\begin{aligned} \int d\Lambda' d\Lambda'' \theta_a(\epsilon') \theta_a(\epsilon'') &= \frac{1}{2} \Lambda_a^2 \\ \int d\Lambda' d\Lambda'' \theta_a(\epsilon') \theta_+(\epsilon'') &= 2\Lambda_a \Lambda_b - \Lambda_b^2 \\ \int d\Lambda' d\Lambda'' \theta_+(\epsilon') \theta_a(\epsilon'') &= \Lambda_b^2 \\ \int d\Lambda' d\Lambda'' \theta_+(\epsilon') \theta_+(\epsilon'') &= 2\Lambda_b^2 \end{aligned} \quad (22)$$

which leads to some simplification of the second-order corrections as predicted by the RG equations

$$\begin{aligned} G''_a &= \Lambda_a^2 (a_1^0 + a_2^0) \frac{\partial (a_1^0 + a_2^0)}{\partial G_a^0} \\ &\quad + (2\Lambda_a \Lambda_b - \Lambda_b^2) a_3^0 \frac{\partial a_2^0}{\partial G_b^0} \\ G''_b &= \Lambda_b^2 \left( (a_1^0 + a_2^0) \frac{\partial a_3^0}{\partial G_a^0} + a_3^0 \frac{\partial a_3^0}{\partial G_b^0} \right) \end{aligned} \quad (23)$$

We find that the similar expressions applicable for the first regime  $V_a \gtrsim V_b$  are obtained from (23) by the replacement  $\Lambda_b \rightarrow \Lambda_a$ . These results may now be compared with the explicit calculation of the second order perturbative corrections, undertaken in the next section.

### B. Perturbation theory in second order

The contribution to the current in the outgoing channel  $j$  at position  $z$  in second order in the interaction may be expressed as

$$J_j^{(2)}(z) = -2 \int \frac{d\Omega d\omega d\omega'}{(2\pi)^2} \int_a^L dx dx' \quad (24)$$

$$\times \sum_{l,l'} \alpha_l \alpha_{l'} T_{jll'}(x, x'; \Omega, \omega, \omega')$$

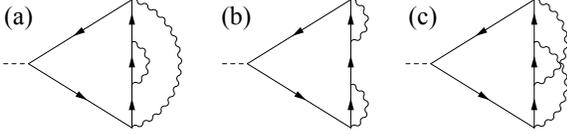


FIG. 2: Three skeleton diagrams showing the leading contribution to currents in second order of perturbation

There are three diagrams (see Fig. 2) contributing in second order, each one with arrows both forming a right-handed or a left-handed loop, which amounts to letting  $\Omega \rightarrow -\Omega$ , giving the identical result, thus leading to a prefactor of 2. The three diagrams give rise to the combination  $T_{jll'}(x, x'; \Omega, \omega, \omega') = T^{(a)} + T^{(b)} + T^{(c)}$ , where

$$T^{(a)} = \sum_{\mu, \mu'=1,2} \sum_{\eta_l, \eta_l'=+,-} \text{Tr}_K [\underline{\gamma}^{ext} \underline{G}_\Omega(j, +, z; l, -\eta_l, x)$$

$$\cdot \bar{\underline{\gamma}}^\mu \underline{G}_{\Omega+\omega}(l, -\eta_l, x; l', -\eta_{l'}, x') \bar{\underline{\gamma}}^{\mu'}$$

$$\cdot \underline{G}_{\Omega+\omega+\omega'}(l', -\eta_{l'}, x'; l', \eta_{l'}, x') \underline{\gamma}^{\mu'}$$

$$\cdot \underline{G}_{\Omega+\omega}(l', \eta_{l'}, x'; l, \eta_l, x) \underline{\gamma}^\mu \underline{G}_\Omega(l, \eta_l, x; j, +, z)] \quad (25)$$

is a rainbow diagram with nested self-energy insertions,

$$T^{(b)} = \sum_{\mu, \mu'=1,2} \sum_{\eta_l, \eta_l'=+,-} \text{Tr}_K [\underline{\gamma}^{ext} \underline{G}_\Omega(j, +, z; l, -\eta_l, x) \bar{\underline{\gamma}}^\mu$$

$$\cdot \underline{G}_{\Omega+\omega}(l, -\eta_l, x; l, \eta_l, x) \underline{\gamma}^\mu \underline{G}_\Omega(l, \eta_l, x; l', -\eta_{l'}, x') \bar{\underline{\gamma}}^{\mu'}$$

$$\cdot \underline{G}_{\Omega+\omega'}(l', -\eta_{l'}, x'; l', \eta_{l'}, x') \underline{\gamma}^{\mu'} \underline{G}_\Omega(l', \eta_{l'}, x'; j, +, z)] \quad (26)$$

is a chain diagram with two self-energy insertions in series, and the third diagram has crossed self-energy insertions

$$T^{(c)} = \sum_{\mu, \mu'=1,2} \sum_{\eta_l, \eta_l'=+,-} \text{Tr}_K [\underline{\gamma}^{ext} \underline{G}_\Omega(j, +, z; l, -\eta_l, x)$$

$$\cdot \bar{\underline{\gamma}}^\mu \underline{G}_{\Omega+\omega}(l, -\eta_l, x; l', -\eta_{l'}, x') \bar{\underline{\gamma}}^{\mu'}$$

$$\cdot \underline{G}_{\Omega+\omega+\omega'}(l', -\eta_{l'}, x'; l, \eta_l, x) \underline{\gamma}^\mu$$

$$\cdot \underline{G}_{\Omega+\omega'}(l, \eta_l, x; l', \eta_{l'}, x') \underline{\gamma}^{\mu'} \underline{G}_\Omega(l', \eta_{l'}, x'; j, +, z)] \quad (27)$$

Here  $\underline{G}_\Omega$  are  $2 \times 2$  matrices of Green's functions in Keldysh space in the absence of interaction, but in the presence of the scattering effect of the junction, which is expressed in terms of the  $S$ -matrix elements  $S_{ij}$ , as presented above. The dependence on the coordinates may be split off:

$$T_{jll'}^{(a,b,c)}(x, x'; \Omega, \omega, \omega') = e^{-2i(\omega x + \omega' x')} T_{jll'}^{(a,b,c)}(\Omega, \omega, \omega') \quad (28)$$

The trace  $\text{Tr}_K$  is over the lower (fermionic) Keldysh indices; the fermion-boson vertices,  $\gamma_{ij}^\mu \rightarrow \underline{\gamma}_{ij}^\mu$ ,  $\bar{\gamma}_{ij}^\mu \rightarrow \bar{\underline{\gamma}}_{ij}^\mu$ , tensors of rank 3 defined in Keldysh space, are given by

$$\gamma_{ij}^1 = \bar{\gamma}_{ij}^2 = \frac{1}{\sqrt{2}} \delta_{ij}, \quad \gamma_{ij}^2 = \bar{\gamma}_{ij}^1 = \frac{1}{\sqrt{2}} \tau_{ij}^1, \quad (29)$$

with  $\tau^1$  the first Pauli matrix. The external vertex is given by

$$\gamma_{ij}^{ext} = \frac{i}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad (30)$$

which suggests to interpret the trace in Keldysh space as operating with the Keldysh matrices on the vector  $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$  and forming the inner product of the resulting vector with the vector  $\frac{i}{2} \begin{pmatrix} 1 & 1 \end{pmatrix}$ .

The calculation of second order corrections to the currents  $J_a, J_b$  is a tedious procedure and is discussed in more detail in the Appendix. Here we provide the summary of this calculation. We find the corrections in the form

$$G_a'' = B_{a1} F_{a1} + B_{a2} F_{a2} + B_{a3} F_{a3},$$

$$G_b'' = B_{b1} F_{b1} + B_{b2} F_{b2} + B_{b3} F_{b3} \quad (31)$$

with

$$\begin{aligned}
B_{a1} &= \frac{1}{16} G_b (G_a - 1 + g_3 (2G_a - 1)) \\
&\quad \times (4G_a + 3G_b - 4 + g_3 (6G_b - 4)), \\
B_{a2} &= -(2G_a - 1) (4G_a^2 - 4G_a + G_b), \\
B_{a3} &= -\frac{1}{8} (1 + 2g_3) G_b (4G_a^2 - 4G_a + G_b), \\
B_{b1} &= \frac{1}{16} G_b (2G_a + G_b - 2 + 2g_3 (G_b - 1)) \\
&\quad \times (4G_a + 3G_b - 4 + g_3 (6G_b - 4)), \\
B_{b2} &= \frac{1}{4} G_b ((1 + 2g_3) G_a G_b - 4(G_a - G_a^2) - g_3 G_b), \\
B_{b3} &= \frac{1}{16} G_b^2 (G_b + 4g_3(1 + g_3) (G_b - 1)).
\end{aligned} \tag{32}$$

Coefficients  $F_j$  are defined as integrals over energy, they are independent of  $G_{a,b}$  and are discussed in the Appendix.

We distinguish again between two regimes: i)  $V_a \gtrsim V_b$  and ii)  $V_a \ll V_b$ . In second regime we find, using the results given in the Appendix

$$\begin{aligned}
F_{a1} &= \Lambda_b^2, & F_{b1} &= 2\Lambda_b^2, \\
F_{a2} &= -\Lambda_a^2, & F_{b2} &= 2\Lambda_b^2, \\
F_{a3} &= \Lambda_b^2 - 4\Lambda_a\Lambda_b, & F_{b3} &= 2\Lambda_b^2.
\end{aligned} \tag{33}$$

which gives

$$\begin{aligned}
G''_a &= B_{a1}\Lambda_b^2 - B_{a2}\Lambda_a^2 + B_{a3}(\Lambda_b^2 - 4\Lambda_a\Lambda_b), \\
G''_b &= 2(B_{b1} + B_{b2} + B_{b3})\Lambda_b^2,
\end{aligned} \tag{34}$$

in the first regime we should merely replace  $\Lambda_b \simeq \Lambda_a$  in these expressions. In both cases one can check the equivalence of (23) and (34), which proves that the second order corrections are indeed exactly generated by the RG equations (18).

Alternatively, we checked the validity of (18) by application of Eq. (10). In the non-trivial second regime  $V_a \ll V_b$ , we use the above expressions, (34) and find that the terms  $\sim \alpha^2$  in (10) are proportional to  $(\Lambda_a - \Lambda_b) d\Lambda_b/d\Lambda$ , which is identically zero.

### C. RG-equations to second order

As shown previously,<sup>23</sup> there are also contributions linear in  $\Lambda$  to the conductances in second order of the interaction, i.e. subleading terms  $\sim \alpha^2\Lambda$ . These contributions arise from the diagram shown in Fig. 3, featuring two fermion loops. Terms linear in  $\Lambda$  generate contributions to the RG beta functions. They give rise to  $\alpha^2$ -corrections to the scaling exponents. The modification of the subleading terms in multi-lead junctions out of equilibrium was not previously analyzed.

Performing calculations similar to the one described in the Appendix we arrive at the following results. The beta

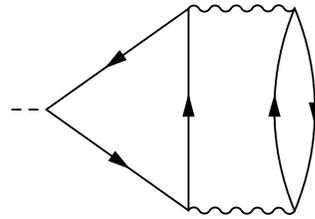


FIG. 3: A skeleton diagram showing the subleading contribution to currents in second order of perturbation

functions in Eq. (18) retain their general structure, with updated coefficient functions  $a_j$ . We have to replace

$$\begin{aligned}
a_1 &\rightarrow (1 - \alpha(G_a - \frac{1}{2})) a_1, \\
a_2 &\rightarrow a_2 + \frac{\alpha}{8} G_b (G_a - \frac{1}{2}) \\
&\quad \times (1 - \frac{1}{4} G_b - g_3(G_b + g_3(G_b - 1))), \\
a_3 &\rightarrow a_3 - \frac{\alpha}{4} G_b \left(1 + 2G_a(G_a - 1) - \frac{3}{4} G_b \right. \\
&\quad \left. + \frac{1}{4}(G_b + 2g_3(G_b - 1))^2\right).
\end{aligned} \tag{35}$$

For the detached third wire,  $G_b = 0$ , we obtain  $a_2(G_a, 0) = a_3(G_a, 0) = 0$  and the modification of  $a_1(G_a, 0)$  is in accordance with the second order expansion of Eq. (47) in Ref.<sup>23</sup>. Notice that for the pure tunneling case,<sup>10</sup> when  $G_b = 4G_a(1 - G_a)$ , the part with  $\theta_a(\epsilon)$  disappears,  $a_1 = 0$ , and the two RG equations with  $\theta_+(\epsilon)$  become linearly dependent, since in this case  $\frac{d}{d\Lambda}(G_b - 4G_a(1 - G_a)) = 0$ .<sup>22</sup>

At the same time there is not much simplification in (35) in the case of  $V_a \sim V_b$ , when we can let  $\theta_+(\epsilon) \simeq 2\theta_a(\epsilon)$ . The remaining expressions are complicated, as can be seen, e.g. by expanding Eq. (12) in<sup>14</sup> in powers of  $\alpha$  at  $c = 0$ .

## V. SUMMARY

In this paper we established the validity of the RG equations for the conductances of multilead junctions of Tomonaga-Luttinger liquid wires in a situation out of equilibrium. Comparing to the equilibrium case, when the RG flow stops at some unique cutoff, which characterizes the low-energy scale of the whole system of wires, the out-of-equilibrium situation can be characterized by several such scales, referring to  $(N - 1)$  relative voltages between the  $N$  wires. In this situation it is not clear which of these scales should be used as a cutoff in the corresponding expressions. Previously we found<sup>22</sup> that the RG equations contained several functions, describing partial stops of the RG flow, so that the direction of the flow could alter during the renormalization process. In this paper we formulate the statement about the scaling property for the set of conductances, characterizing a general setup with  $N$  wires. Then we consider the particular example of the Y junction ( $N = 3$ ) with different

strength of interaction in the main wire and the tunneling tip. We focus on the two most interesting regimes, when i) all voltages are of the same order and ii) the voltage  $V_a$  in the main wire is much smaller than the voltage  $V_b$  at the tip.

The second order corrections are calculated in two ways. One way is the iteration of the RG equations to second order, which is less trivial in the presence of several cutoffs. A second way is the direct calculation of second order corrections by means of computer algebra, which requires considering a large number of partly canceling contributions. We find that both ways of calculation lead to identical results in both regimes. As a by-product we derived the corrections to the beta functions of second order in the interaction.

We believe that our results may be useful for a generalization of ideas of scaling in the presence of several low energy cutoffs, appearing particularly in out-of-equilibrium situations.

## VI. ACKNOWLEDGEMENTS

The work of D.A. was partly supported by RFBR grant No. 15-52-06009 .

### Appendix A: Details of calculation

The expression for the corrections (24) requires five integrations.

Let us first discuss the integration over  $x, x'$ . The dependence of each Green's function in (3) on the coordinates comes from two factors: the step functions,  $\theta(x)$ , and the oscillatory exponentials,  $e^{i\omega x}$ . The outgoing current is determined at a point  $z$  in the lead, which is outside the interacting region. In our terms this means that the coordinate  $z$  is greater than any other of the coordinates,  $x, x'$ . This allows to simplify the step functions by replacing  $\theta(z-x) = 1$ ,  $\theta(x-z) = 0$ , etc. The corrections to the incoming currents are zero, which is verified by putting  $\theta(z-x) = 0$ ,  $\theta(x-z) = 1$ . The exponents  $e^{\pm i\omega x}$  do not contain  $\Omega$ , and after appropriate change of sign in  $\omega$ ,  $\omega'$  can be reduced to unique form  $e^{-2i(\omega x + \omega' x')}$ . The remaining expressions may still contain  $\theta(x-x')$ ,  $\theta(x'-x)$ , however, after symmetrization,  $x \leftrightarrow x'$ ,  $\omega \leftrightarrow \omega'$ , these stepwise functions combine to unity.

The integration over  $x, x'$  is now simple, since the dependence on the coordinates in each term is reduced to  $e^{-2i(\omega x + \omega' x')}$ . We have

$$\int_a^L dx_1 e^{-2i\omega x} = \frac{e^{-2i\omega a} - e^{-2i\omega L}}{2i\omega} \rightarrow \frac{1}{2i\omega}$$

where the last equality is obtained because the rapidly oscillating factor  $e^{-2i\omega L}$  is only important as an infrared cutoff at the smallest  $\omega$ , and in our case this cutoff is provided by the voltages. The integration over  $x, x'$  hence leads to the overall factor,  $-1/(4\omega_1\omega_2)$ .

It is convenient to symmetrize the appearing expressions with respect to  $\omega \rightarrow -\omega$ , picking the odd-in- $\omega$  part of the integrand, and then to consider a positive interval of energies in subsequent integrations :

$$\int_0^{\omega_0} \frac{d\omega d\omega'}{4\omega\omega'} \quad (\text{A1})$$

with  $\omega_0$  ultraviolet cutoff.

Let us now discuss the integration over  $\Omega$ . In general, we find terms, linear in  $h_l(\omega) = \tanh[(\omega - \mu_l)/2T] \equiv h_0(\omega - \mu_l)$ , and cubic in this quantity,  $\sim h_l h_m h_j$ . The quadratic terms,  $h_l h_m$ , disappear.

Every cubic combination has the form  $h_0(\Omega_1)h_0(\Omega_2)h_0(\Omega_3)$ , with  $\Omega_j = \Omega + \dots$ , (e.g.  $\Omega_j = \Omega - \mu_1 + \omega$  or  $\Omega_j = \Omega - \mu_2 + \omega - \omega'$ ). In order to regularize the integral over  $\Omega$ , we subtract a term  $h_0(\Omega_3)$  so that the combination  $(h_0(\Omega_1)h_0(\Omega_2) - 1)h_0(\Omega_3)$  is convergent at  $\Omega \rightarrow \pm\infty$ . All regularization terms  $h_0(\Omega_3)$  are combined with the other terms of the first power in  $h_0(\dots)$ .

In the so regularized terms we may shift the argument and write

$$\begin{aligned} & \int d\Omega (h_0(\Omega + \tilde{\omega}_1)h_0(\Omega + \tilde{\omega}_2) - 1)h_0(\Omega + \tilde{\omega}_3), \\ &= \int d\Omega (h_0(\Omega + \tilde{\omega}_1 - \tilde{\omega}_3)h_0(\Omega + \tilde{\omega}_2 - \tilde{\omega}_3) - 1)h_0(\Omega), \\ &\equiv f_3(\tilde{\omega}_1 - \tilde{\omega}_3, \tilde{\omega}_2 - \tilde{\omega}_3) \end{aligned} \quad (\text{A2})$$

with

$$\begin{aligned} f_3(A, B) &= \int d\Omega \tanh \frac{\Omega}{2T} (\tanh \frac{\Omega+A}{2T} \tanh \frac{\Omega+B}{2T} - 1), \\ &= 2 \coth \frac{A-B}{2T} (A \coth \frac{A}{2T} - B \coth \frac{B}{2T}), \\ &\rightarrow 2(|A| - |B|) \text{sign}|A - B|, \end{aligned} \quad (\text{A3})$$

the last line obtained in the limit  $T \rightarrow 0$ , which we are mostly interested in.

The terms linear in  $h_0(\dots)$  cancel each other. This can be proved in two steps. First we subtract one and the same term  $h_0(\Omega)$  from each term,  $h_0(\Omega + \tilde{\omega}_1) \rightarrow h_0(\Omega + \tilde{\omega}_1) - h_0(\Omega)$ , making the integral convergent:

$$\int d\Omega (h_0(\Omega + \tilde{\omega}_1) - h_0(\Omega)) = -2\tilde{\omega}_1. \quad (\text{A4})$$

The combination of all such terms does not contain  $\omega, \omega'$  and hence vanishes when performing the symmetrization  $\omega \rightarrow -\omega$ , leading to above expression (A1). As a second step we sum all terms with  $h_0(\Omega)$  and verify that they cancel each other.

Let us discuss further simplifications. When obtaining the terms,  $f_3(\dots)$ , the third argument  $\Omega + \tilde{\omega}_3$  in Eq. (A2) was chosen by computer, i.e. almost randomly from the human viewpoint. It means that many appearing terms may look differently but lead to the same result after subsequent integrations. To get rid of this ambiguity we

use the symmetry properties

$$\begin{aligned} \text{(i)} \quad & f_3(a, b) = f_3(b, a), \quad \text{(ii)} \quad f_3(a, b) = -f_3(-a, -b), \\ \text{(iii)} \quad & f_3(a, b) \approx f_3(a - b, -b), \end{aligned} \quad (\text{A5})$$

where the last approximate equality means generation of the linear in  $h_0$  terms which eventually disappear. The last equivalence property is (iv) the symmetry with respect to  $\omega \leftrightarrow \omega'$ .

We find that the number of expressions to be considered is strongly reduced, when we take each correction term of the form  $Af_3(a, b)$ , strip its prefactor  $A$  and perform the symmetry operations, (i) – (iv), for  $f_3(a, b)$ . We thus form an equivalence list of length  $2^4 = 16$  which is ordered according to the computer's internal rules. We choose a first element  $f_3(a_*, b_*)$  of this ordered list as a representative. Such operation leaves only the factors  $f_3(a_*, b_*)$  which are not related by symmetry operations, i.e. are essentially different.

Note that at this step of our analysis we may find terms of the form  $f_3(\omega' - \omega + a, b)$  with  $a, b$  dependent only on the  $\mu_j$ . Recalling the initial expression  $e^{-2i(\omega x + \omega' x')}$  we see that a shift  $\omega' = \bar{\omega} + \omega$  and integration over  $\omega$  leads to  $\delta(x + x')$ . In combination with the condition  $x, x' > 0$  this means that such terms should be discarded.

In the intermediate expressions for the three types of diagrams in Fig. 2 we found terms containing neither  $\mu_1$  nor  $\mu_2$ . Such unphysical terms cancel in the combination of all three types of diagrams.

To condense the expressions further we introduce the symmetric combination

$$\begin{aligned} f_s(A, B) = & -f_3(-\omega - A, \omega' - B) + f_3(-\omega + A, \omega' + B) \\ & + f_3(\omega - A, \omega' - B) - f_3(\omega + A, \omega' + B) \end{aligned} \quad (\text{A6})$$

with the properties

$$\begin{aligned} f_s(A, B) & \simeq 8A, \quad \omega \ll \omega', \\ & \simeq 8B, \quad \omega \gg \omega', \end{aligned} \quad (\text{A7})$$

$$f_s(A, B)|_{\omega=0} = f_s(A, B)|_{\omega'=0} = 0$$

From the piecewise linear form of (A3) it is clear that the transition between the values  $f_s \simeq 8A$  and  $f_s \simeq 8B$  takes place at  $|\omega - \omega'| \lesssim |A - B|$ .

The corrections to the currents are then expressed through integrals

$$\begin{aligned} & \{F_{a1}, F_{a2}, F_{a3}, F_{b1}, F_{b2}, F_{b3}\} \\ = & - \int_0^{\omega_0} \frac{d\omega d\omega'}{4\omega\omega'} \left\{ f_s(\mu_1, 0) - f_s(\mu_2, 0), \right. \\ & f_s(\mu_2 - \mu_1, 0), \\ & f_s(\mu_2 - \mu_1, \mu_2) + f_s(\mu_2 - \mu_1, -\mu_1), \\ & f_s(\mu_1, 0) + f_s(\mu_2, 0), f_s(\mu_1, \mu_2), \\ & \left. f_s(\mu_2 - \mu_1, \mu_2) - f_s(\mu_2 - \mu_1, -\mu_1) \right\} \end{aligned} \quad (\text{A8})$$

We have here a generic integral

$$\begin{aligned} & \int_0^{\omega_0} \frac{d\omega d\omega'}{4\omega\omega'} f_s(A, B) \\ = & A \ln^2 \frac{\omega_0}{|A|} + B \ln^2 \frac{\omega_0}{|B|}, \quad |A| \sim |B|, \\ = & (A - B) \ln^2 \frac{\omega_0}{|A|} + 2B \ln \frac{\omega_0}{|A|} \ln \frac{\omega_0}{|B|}, \quad |A| \gg |B|, \end{aligned} \quad (\text{A9})$$

we use these estimates for the calculation of the results (33) in the main text.

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